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ON EMBEDDABILITY AND STRESSES OF GRAPHS

ERAN NEVO*

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Gluck has proven that triangulated 2-spheres are generically 3-rigid. Equivalently, planar graphs are generically 3-stress free. We show that already the K_5-minor freeness guarantees the stress freeness. More generally, we prove that every K_{r+2}-minor free graph is generically r-stress free for $1 \le r \le 4$. (This assertion is false for $r \ge 6$.) Some further extensions are discussed.

1. Introduction and results

Gluck [6] has proven that triangulated 2-spheres are generically 3-rigid. His proof is based on two classical theorems: one is Cauchy's rigidity theorem [3] (for convex 3-polytopes), the other is Steinitz's theorem [17] which asserts that any polyhedral 2-sphere is combinatorially isomorphic to the boundary complex of some convex 3-polytope. Whiteley [18] has found a proof of Gluck's theorem which avoids convexity, based on vertex splitting. This proof inspired our main result, Theorem 1.2 below.

It is easy to see that a graph with n vertices and $3n-6$ edges is generically 3-rigid iff it is generically 3-stress free. Thus, Gluck's theorem can be stated as:

Theorem 1.1 (Gluck). *Planar graphs are generically 3-stress free.*

Bearing in mind Kuratowski's theorem [9], we show that already the K_5-minor freeness guarantees the 3-stress freeness, and more generally:

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Theorem 1.2. *For $2 \le r \le 6$, every K_r-minor free graph is generically $(r-2)$-stress free.*

The proof is based on contracting edges which possess a certain property. It makes an essential use of Mader's theorem [12] which gives an upper bound $(r-2)n-\binom{r-1}{2}$ on the number of edges in a K_r-minor free graph with n vertices, for $r \le 7$. Indeed, Theorem 1.2 can be regarded as a strengthening of Mader's theorem, as being generically l-stress free implies having at most $ln-\binom{l+1}{2}$ edges (as the rank of the corresponding rigidity matrix is at most $ln-\binom{l+1}{2}$, see Section 2). This also shows that Theorem 1.2 fails for $r \ge 8$, as is demonstrated for $r=8$ by $K_{2,2,2,2,2}$, and for $r>8$ by repeatedly coning over $K_{2,2,2,2,2}$ (e.g. [16]). It would be interesting to find a proof of Theorem 1.2 that avoids using Mader's theorem (and derive Mader's theorem as a corollary).

Problem 1.3. Must a graph with a generic $(r-2)$-stress contain a subdivision of K_r for $2 \le r \le 6$?

The answer is positive for $r=2,3,4$ as in this case G has a K_r minor iff G contains a subdivision of K_r ([5], Proposition 1.7.2). Mader proved that every graph on n vertices with more than $3n-6$ edges contains a subdivision of K_5 [13]. A positive answer in the case $r=5$ would strengthen this result.

A graph is *linklessly embeddable* if there exists an embedding of it in $\mathbb{R}^3$ (where vertices and edges have disjoint images) such that every two disjoint cycles of it are unlinked closed curves in $\mathbb{R}^3$. As such graph is K_6-minor free (e.g. [15], [11]), combining with Theorem 1.2 we conclude:

Corollary 1.4. *Linklessly embeddable graphs are generically 4-stress free.*

Let $\mu(G)$ denote the Colin de Verdière's parameter of a graph G [4]. Colin de Verdière [4] proved that a graph G is planar iff $\mu(G) \le 3$; Lovász and Schrijver [11] proved that G is linklessly embeddable iff $\mu(G) \le 4$. While we have seen that Theorem 1.2 fails for $r \ge 8$, we conjecture that Theorem 1.1 and Corollary 1.4 extend to:

Conjecture 1.5. Let G be a graph and let k be a positive integer. If $\mu(G) \le k$ then G is generically k-stress free.

For $k=1,2,3,4$ this is true: Colin de Verdière [4] showed that the family $\{G : \mu(G) \le k\}$ is closed under taking minors for every k. Note that $\mu(K_r)=r-1$. By Theorem 1.2 Conjecture 1.5 holds for $k \le 4$. Conjecture 1.5 implies

$$\mu(G) \le k \Rightarrow e \le kv - \binom{k+1}{2}$$

(where e and v are the numbers of edges and vertices in G, respectively) which is not known either.

This paper is organized as follows: Section 2 provides relevant background in rigidity theory of graphs, Section 3 deals with graph minors, and in Section 4 we prove Theorem 1.2.

2. Rigidity

The presentation here is based mainly on Kalai's [7]. Let $G=(V,E)$ be a graph. Let $d(a,b)$ denote Euclidian distance between points a and b in Euclidian space. A d-embedding $f: V \to \mathbb{R}^d$ is called *rigid* if there exists an $\varepsilon > 0$ such that if $g: V \to \mathbb{R}^d$ satisfies $d(f(v),g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u),g(w)) = d(f(u),f(w))$ for every $\{u,w\} \in E$, then $d(g(u),g(w)) = d(f(u),f(w))$ for every $u,w \in V$. Loosely speaking, f is rigid if any perturbation of it which preserves the lengths of the edges is induced by an isometry of $\mathbb{R}^d$. G is called *generically d-rigid* if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings. Given a d-embedding $f: V \to \mathbb{R}^d$, a *stress* w.r.t. f is a function $w: E \to \mathbb{R}$ such that for every vertex $v \in V$

$$\sum_{u:\{v,u\}\in E} w(\{v,u\})(f(v)-f(u)) = 0.$$

G is called *generically d-stress free* if the set of its d-embeddings which have a unique stress ($w=0$) is open and dense in the space of all of its d-embeddings.

Rigidity and stress freeness can be related as follows: Let $V=[n]$, and let $\mathrm{Rig}(G,f)$ be the $dn \times |E|$ matrix associated with a d-embedding f of $V(G)$ defined as follows: for its column corresponding to $\{v<u\} \in E$ put the vector $f(v)-f(u)$ (resp. $f(u)-f(v)$) at the entries of the rows corresponding to v (resp. u) and zero otherwise. G is generically d-stress free if $\mathrm{Ker}(\mathrm{Rig}(G,f))=0$ for a generic f (i.e. for an open dense set of embeddings). G is generically d-rigid if $\mathrm{Im}(\mathrm{Rig}(G,f)) = \mathrm{Im}(\mathrm{Rig}(K_V,f))$ for a generic f, where K_V is the complete graph on $V=V(G)$. The dimensions of the kernel and image of $\mathrm{Rig}(G,f)$ are independent of the generic f we choose; we call $R(G)=\mathrm{Rig}(G,f)$ the *rigidity matrix* of G. For the complete graph, one computes $\mathrm{rank}(\mathrm{Rig}(K_V)) = dn - \binom{d+1}{2}$ (see Asimov and Roth [1] for more details).

We need the following two known results:

Theorem 2.1 (Asimov and Roth [2]). *Let $G_i=(V_i,E_i)$ be generically k-stress free graphs, $i=1,2$ such that $G_1\cap G_2$ is generically k-rigid. Then $G_1\cup G_2$ is generically k-stress free.*

Theorem 2.2 (Whiteley [18]). *Let G' be obtained from a graph G by contracting an edge $\{u,v\}$.*

(a) If u,v have at least $d-1$ common neighbors and G' is generically d-rigid, then G is generically d-rigid.

(b) If u,v have at most $d-1$ common neighbors and G' is generically d-stress free, then G is generically d-stress free.

Theorem 2.2 gives an alternative proof of Gluck's theorem (Whiteley [18]): starting with a triangulated 2-sphere, repeatedly contract edges with exactly 2 common neighbors until a tetrahedron is reached (it is not difficult to show that this is always possible). By Theorem 2.2(a) it is enough to show that the tetrahedron is generically 3-rigid, as is well known (Asimov and Roth [1]).

Remark. Generic rigidity is equivalent to a certain property of Kalai's symmetric algebraic shifting of the graph, see [8] and [10] for details. Kalai defined exterior shifting as well; see [8] for details and references. Let Δ denote the algebraic shifting operator, for both symmetric and exterior versions. Theorem 1.2 extends to:

for every $2\le r\le 6$ and every graph G, if $\{r-1,r\}\in\Delta(G)$ then G has a K_r minor.

Details can be found in early version of this paper [14]. In relation with Conjecture 1.5, we conjecture $\mu(G)\le k\Rightarrow\{k+1,k+2\}\notin\Delta(K)$. The early version [14] also discusses connections with embeddability into 2-manifolds.

3. Minors

All graphs we consider are simple, i.e. with no loops and no multiple edges. Let $e=\{v,u\}$ be an edge in a graph G. By *contracting* e we mean identifying the vertices v and u and deleting the loop and one copy of each double edge created by this identification, to obtain a new (simple) graph. A graph H is called a *minor* of a graph G, denoted $H\prec G$, if by repeated contraction of edges we can obtain H from a subgraph of G. In the sequel we shall make an essential use of the following Theorem of Mader:

Theorem 3.1 (Mader [12]). *For $3\le r\le 7$, if a graph G on n vertices has no K_r minor then it has at most $(r-2)n-\binom{r-1}{2}$ edges.*

Proposition 3.2. *For $3 \leq r \leq 5$: If G has an edge and each edge belongs to at least $r-2$ triangles, then G has a K_r minor.*

Proof. For $r=3$ G actually contains K_3 as a subgraph. Let G have n vertices and e edges. Assume (by contradiction) that $K_r \not< G$. W.l.o.g. G is connected.

For $r=4$, by Theorem 3.1 $e \leq 2n-3$ hence there is a vertex $u \in G$ with degree $d(u) \leq 3$. Denote by $N(u)$ the induced subgraph on the neighbors of u. For every $v \in N(u)$, the edge uv belongs to at least two triangles, hence $N(u)$ is a triangle, and together with u we obtain a K_4 as a subgraph of G, a contradiction.

For $r=5$, by Theorem 3.1 $e \leq 3n-6$ hence there is a vertex $u \in G$ with degree $d(u) \leq 5$. Also $d(u) \geq 4$ (as we may assume that u is not an isolated vertex). If $d(u)=4$ then the induced subgraph on $\{u\} \cup N(u)$ is K_5, a contradiction. Otherwise, $d(u)=5$. Every $v \in N(u)$ has degree at least 3 in $N(u)$, hence $e(N(u)) \geq \lceil 3 \cdot 5/2 \rceil = 8$. But $K_4 \not< N(u)$, hence $e(N(u)) \leq 2 \cdot 5 - 3 = 7$, a contradiction. ∎

Proposition 3.3. *If G has an edge and each edge belongs to at least 4 triangles, then either G has a K_6 minor, or G is a clique sum over K_r for some $r \leq 4$ (i.e. $G = G_1 \cup G_2, G_1 \cap G_2 = K_r$, $G_i \neq K_r$, $i=1,2$).*

Proof. We proceed as in the proof of Proposition 3.2: Assume that $K_6 \not< G$. W.l.o.g. G is connected. By Theorem 3.1 $e \leq 4n-10$ hence there is a vertex $u \in G$ with degree $d(u) \leq 7$, also $d(u) \geq 5$. If $d(u)=5$ then $N(u)=K_5$, a contradiction. Actually, $N(u)$ is planar: since $N(u)$ has at most 7 vertices, each of degree at least 4, if $N(u)$ were not 4-connected, it must have exactly 7 vertices and two disjoint edges such that each of their 4 vertices is adjacent to the remaining 3 vertices of $N(u)$ (whose removal disconnect $N(u)$); but such graph has a K_5 minor. As $K_5 \not< N(u)$, $N(u)$ is 4-connected. Now Wagner's structure theorem for K_5-minor free graphs ([5], Theorem 8.3.4) asserts that $N(u)$ is planar.

If $d(u)=6$, then $12 = 3 \cdot 6 - 6 \geq e(N(u)) \geq 4 \cdot 6/2 = 12$ hence $N(u)$ is a triangulation of the 2-sphere S^2. If $d(u)=7$, then $15 = 3 \cdot 7 - 6 \geq e(N(u)) \geq 4 \cdot 7/2 = 14$. We will show now that $N(u)$ cannot have 14 edges, hence it is a triangulation of S^2: Assume that $N(u)$ has 14 edges, so each of its vertices has degree 4, and $N(u)$ is a triangulation of S^2 minus an edge. Let us look to the unique square (in a planar embedding) and denote its vertices by A. The number of edges between A and $N(u)-A$ is 8. Together with the 4 edges in the subgraph induced by A, leaves two edges for the subgraph induced by $N(u)-A=\{a,b,c\}$; let a be their common vertex. We now look at the neighborhood of a in a planar embedding (it is a 4-cycle): b,c must be opposite in this square as $\{b,c\}$ is missing. Hence for $v \in A \cap N(a)$ we get that v has degree 5, a contradiction.

Now we are left to deal with the case where $N(u)$ is a triangulation of S^2, and hence a maximal K_5-minor free graph. If G is the cone over $N(u)$ with apex u, then every edge in $N(u)$ belongs to at least 3 triangles in $N(u)$. By Proposition 3.2, $N(u)$ has a K_5 minor, a contradiction. Hence there exists a vertex $w \neq u$, $w \in G \backslash N(u)$. Denote by $[w]$ the set of all vertices in G connected to w by a path disjoint from $N(u)$. Denote by $N'(w)$ the induced graph on the vertices in $N(u)$ that are neighbors of some vertex in $[w]$. If $N'(w)$ is not a clique, there are two non-neighbors $x, y \in N'(w)$, and a path through vertices of $[w]$ connecting them. This path together with the cone over $N(u)$ with apex u form a subgraph of G with a K_6 minor, a contradiction.

Suppose $N'(w)$ is a clique (it has at most 4 vertices, as $N(u)$ is planar). Then G is a clique sum of two graphs that strictly contain $N'(w)$: Let G_1 be the induced graph on $[w] \cup N'(w)$ and let G_2 be the induced graph on $G \backslash [w]$. Then $G = G_1 \cup G_2$ and $G_1 \cap G_2 = N'(w)$. ∎

Remark. In view of Theorem 3.1 for the case $r = 7$, we may expect the following to be true:

Problem 3.4. If G has an edge and each edge belongs to at least 5 triangles, then either G has a K_7 minor, or G is a clique sum over K_l for some $l \leq 6$.

If true, it extends the assertion of Theorem 1.2 to the case $r = 7$. By now we can show only the weaker assertion

$$G \text{ has a generic 5-stress} \Rightarrow K_7^- \prec G,$$

by using similar arguments to those used for proving Theorem 1.2 (K_7^- is K_7 minus an edge).

4. Proof of Theorem 1.2

For $r = 2$ the assertion of the theorem is trivial. Suppose $K_r \nprec G$, and contract edges belonging to at most $r-3$ triangles as long as it is possible. Denote the resulted graph by G'. Repeated application of Theorem 2.2 asserts that if G' is generically $(r-2)$-stress free, then so is G. In case G' has no edges, it is trivially $(r-2)$-stress free. Otherwise, G' has an edge, and each edge belongs to at least $r-2$ triangles. For $2 < r < 6$, by Proposition 3.2 G' has a K_r minor, hence so has G, a contradiction. For $r = 6$, by Proposition 3.3 G' either has a K_6 minor which leads to a contradiction, or G' is a clique sum over K_r for some $r \leq 4$. In the latter case, denote $G' = G_1 \cup G_2$, $G_1 \cap G_2 = K_r$.

As the graph of a simplex is k-rigid for any k, by Theorem 2.1 it is enough to show that each G_i is generically $(r-2)$-stress free, which follows from induction hypothesis on the number of vertices. ∎

Remark. We can prove the case $r=5$ avoiding Mader's theorem, by using Wagner's structure theorem for K_5-minor free graphs ([5], Theorem 8.3.4) and Theorem 2.1. Using Wagner's structure theorem for $K_{3,3}$-minor free graphs ([5], ex.20 on p.185) and Theorem 2.1, we conclude that $K_{3,3}$-minor free graphs are generically 4-stress free.

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Eran Nevo

Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem
Israel
eranevo@math.huji.ac.il